#4 Suppose we have three positive integers $n-1, n, n+1$ such that $k=n(n-1)(n+1)$ is a square. ID: 346

Lemma 1: For all positive integers m, $GCD(m, m+1)=1$.

Proof 1: If $a \neq 1$ and $a|m$, then $m \equiv 0 \pmod{a}$. This means that $m+1 \equiv 1 \pmod{a}$. Thus, if $a|m$, then $a \nmid (m+1)$, so m and $m+1$ share no common divisors outside of 1. $\square$

By Lemma 1, $GCD(n-1, n) = GCD(n, n+1) = 1$. Thus, any prime that occurs in the prime factorization of n cannot occur in those of $n-1$ and $n+1$.

But $k = n(n+1)(n-1)$ is a square. This means that the prime factorization of k contains only even powers of primes. Since n shares no prime factors with $n-1$ or $n+1$, n must also be a square.

Lemma 2: If $n > 1$ is a positive integer, n^2-1 is not a square.

Proof 2: The largest perfect square below n^2 is $(n-1)^2$.

$(n-1)^2 = n^2-2n+1 = n^2-1+(2-2n)$

If $n>1$, $2-2n<0$. Thus, n^2-1 is larger than the largest square below n^2, so n^2-1 is not a perfect square. $\square$

Notice that $k = n(n^2-1)$. n is a square and n^2-1 is not a square (by Lemma 2). Thus, k is not a perfect square, so there is no set of 3 consecutive positive integers whose product is a square. $\blacksquare$